Indian Statistical Institute, Bangalore Centre

M.Math II Year, First Semester

Semestral Examination

Fourier Analysis

Time: 3 Hours December 5, 2012

Maximum marks you can get is 35.

Maximum marks you can get in Part A is 30; Maximum marks you can get in Part B is 5.

Part A

1. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic satisfying

   (a) $|f(z)| \leq c\, e^{A|z|}$ for some $c, A > 0$, and

   (b) $\sup\limits_{y} \int |f(x+iy)|^2 \, dx < \infty$

   Then show that $f(z) = 0$ for all z. [4]

2. Let B_n be the spline of order n given by

$$B_n(x) = \{\chi_{[0,1]} * \chi_{[0,1]} \ldots * \chi_{[0,1]}\}(x)$$

   where $n+1$ functions are in the RHS. Show that $B_n(x)$ is a finite linear combination of the functions $B_n(2x-k), k = 0, 1, 2, 3, \ldots$. [4]

3. Let H be the real Hilbert transform given by

$$(Hf)(x) = CPV \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \, dy.$$

   Calculate $H\chi_{[a,b]}$ for $-\infty < a < b < \infty$. [6]

4. Let Y be a closed linear subspace of $L^1(R^n)$. Assume that $Y * L^1(R^n) \subset Y$. If $f \in Y$, then show that $\tau_s f \in Y$ where $(\tau_s f)(x) = f(x-s)$. [3]

5. Let $f_k(x) = x^k \, e^{-x^2/2}$ for $k = 0, 1, 2, \ldots$ and x in R. Find a relation between $\hat{f}_{k+1}(\xi), \xi \; \hat{f}_k(\xi)$ and $\hat{f}_{k-1}(\xi)$. [3]

6. (a) Let $f : R \to R$ be in $L^1_{\text{loc}}(R)$ with compact support. If the maximal function Mf is in $L^1(R)$ show that $f = 0$. [4]

   (b) Let $g \in L^1(R)$. If $Mg \in L^1(R)$, then show that $g = 0$. Note that g is not assumed to have compact support. [2]

7. State and prove Marcinkiewicz Theorem. [8]

8. State and prove Heisenberg's uncertainity principle for f in $S(R)$, the Schwartz class on R. [4]

9. State and prove Poisson summation formula. [4]

10. Let $\psi \in L^2(R)$ satisfy $\int dw \frac{|\hat{\psi}(w)|^2}{|w|} < \infty$. Define $\psi_{a,b}$ for $a > 0$ and b in R by $\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\ \psi(\frac{t-b}{a})$. For f in $L^1 \cap L^2(R)$, define $(Wf)(a,b) = \langle f, \psi_{a,b} \rangle$.

    Show that

$$\int_0^\infty \frac{da}{a^2} \int_R db\ |(Wf)(a,b)|^2 = C\|f\|_{L^2}^2$$

    for a constant C depending only on ψ. [6]

Part B

11. Let $f_0(t) = e^t\ \chi_{[-\infty,0]}(t)$ and $f_1(t) = e^{-t}\ \chi_{[0,\infty)}(t)$. For each complex number a define the $L^1(R)$ function g_a by $g_a(t) = f_0(t) + af_1(t)$. Let

    $B = \{a \text{ in } \mathbb{C} : \text{ linear span of translates of } g_a \text{ is not dense in } L^1(R)\}$.

    If $a \in B$ find a relation between $a, \bar{a}$ and 1. Determine B. [5]